Weak S-Normality and Generalized Ordered Spaces

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Abstract

In this paper, we define a new notion "weak S-normality" of a topological space. It generalizes two properties "weak perfectness" and "S-normality" that were studied in our earlier papers. The advantage of taking this notion into account is to widen the category of spaces related to the perfect spaces. When we discuss this property of generalized ordered spaces, we obtain various interesting and important examples that are useful for the studies of weakly perfect spaces and S-normal spaces. From this point of view, we study the weak S-normality of generalized ordered spaces and their linearly ordered extensions.

Key words: linearly ordered space (LOTS), generalized ordered space (GO-space), linearly ordered extension, S-normal space, weakly S-normal space.

1 Definitions and basic results

We review in this section the definition of S_δ-set. This definition is a generalization of a G_δ-set (a G_δ-set is an intersection of countably many open sets) and was introduced by H. R. Bennett [1] to study some properties of LOTS, where LOTS is used as the abbreviation of "linearly ordered (topological) space." See Section 2 for the definitions of LOTS and GO-spaces.

Definition 1.1 Let X be a topological space. A subset A of X is called an *S_δ-set* if there exists a collection $\{U(1), U(2), \cdots\}$ of countably many open subsets of X such that, for two points $p \in A$ and $q \in X \setminus A$, there exists an $n \in N$ such that $p \in U(n)$ and $q \notin U(n)$. The collection mentioned is usually written by $\{U(i) : i \in N\}$, where N denotes the set of natural numbers.

It is easy to see that a G_δ-set is an S_δ-set. Hence S_δ-sets are a generalization of G_δ-sets.

Definition 1.2 Let X be a topological space. X is *perfect* or a *perfect space* if every closed subset of X is a G_δ-set.

As is well known, the Sorgenfrey line is a perfect space. This space is denoted by S. The set $S \times \{-1, 0\}$ equipped with the usual lexicographic order is also a perfect space. This space is identified with the linearly ordered extension $L(S)$ of S that contains S as a dense subspace. See Section 2.

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Definition 1.3 Let X be a topological space. X is *weakly perfect* or a *weakly perfect space* if every closed subset C of X has a dense subset D of C that is a G_δ-set.

As is shown in [3], the space of all countable ordinals is weakly perfect and is denoted by ω_1 or $[0, \omega_1[$. See [3] for further studies on weakly perfect GO-spaces.

Definition 1.4 Let X be a topological space. X is *S-normal* or an *S-normal space* if every closed subset of X is an S_δ-set.

The Michael line is an S-normal space and is denoted by M. To see this, Let C be a closed subset of M. For a point of $x \in C \cap \mathbb{Q}$, where $\mathbb{Q}$ denotes the set of rational numbers, take a collection $\mathcal{U}(x) = \{U_i(x) : i \in N\}$ of open subsets of M such that $\mathcal{U}(x)$ is a neighborhood base at x. Then a collection $\cup\{\mathcal{U}(x) : x \in C \cap \mathbb{Q}\} \cup \{C \cap \mathbb{P}\}$ guarantees for C to be an S_δ-subset of M, where $\mathbb{P}$ denotes the set of irrational numbers. Let $M^* = M \times \{0\} \cup \mathbb{P} \times \mathbb{Z}$ be a linearly ordered extension of M, where $\mathbb{Z}$ denotes the set of integers. See Section 2 for the definition of X^*, where X is a GO-space. It is analogously proved that M^* is an S-normal space.

See [2], [4], [6] for further studies on S-normal GO-spaces.

The following concept is a natural generalization of Definitions 1.2-1.4 that takes the main position of this paper.

Definition 1.5 Let X be a topological space. X is *weakly S-normal* or a *weakly S-normal space* if every closed subset C of X has a dense subset D of C that is an S_δ-subset of X.

Let S be the Sorgenfrey line. It is easy to see that $S^* = S \times \{0, -1, -2, \cdots\}$ is weakly S-normal since $\mathbb{Q}$ is dense in S and $S^* \setminus (S \times \{0\})$ is an open subset. However, S^* is not S-normal, because points $(x, 0)$ and $(x, -1)$, $x \in \mathbb{P}$, can not be separated by countably many open sets. It should be noted that $L(M)$ is also an example of weakly S-normal space, where $L(M) = M \times \{0\} \cup \mathbb{P} \times \{-1, 1\}$ is the linear extension defined in Section 2.

Let $[0, \omega_1] = [0, \omega_1[\cup\{\omega_1\} = \omega_1 \cup \{\omega_1\}$ be the union of the set of all countable ordinals and the least uncoutable ordinal ω_1. Then $[0, \omega_1]$ is a LOTS and is not weakly S-normal. This is shown by the following proposition, because $[0, \omega_1]$ is not first countable at ω_1.

Proposition 1.6 *Let X be a weakly S-normal generalized ordered space. Then X is first countable. (For the definition of generalized ordered spaces or GO-spaces, see Section 2.)*

The proof of this proposition is as same as the cases of S-normality [2] and weak perfectness [3] of generalized ordered spaces.

Proposition 1.7 *The relationships between these notions 1.2-1.5 are as follows. Any converse of the following implications does not hold.*
(1) *A perfect space is weakly perfect. A weakly perfect space is weakly S-normal.*
(2) *A perfect space is S-normal. An S-normal space is weakly S-normal.*

Proof. The implications in (1) and (2) are clear from the definitions. We give couter-examples. Let ω_1 be the linearly ordered space of all countable ordinals. Then ω_1 is a weakly perfect space as is shown in [3], but is not perfect since the closed subset consisting of limit ordinals in ω_1 is not a G_δ-set. The linearly ordered extensions S^* and $L(M)$ are weakly S-normal, but not weakly perfect nor S-normal. The Michael line M and the linearly ordered extension M^* are S-normal spaces, but are not perfect.

2 Two linearly ordered extensions and notation

A linearly ordered space (LOTS) is a linearly ordered set with the order topology λ. A generalized ordered space (GO-space) is a subspace of a LOTS. For a generalized ordered space (X, τ), that is, τ denotes the topology on X containing λ and each point of X has a neighborhood base consisting of convex sets (with respect to the order, and possibly degenerate sets), there are two linearly ordered extensions. One of them is X^* and was defined by D. J. Lutzer [8]. The other one is $L(X)$ and was studied in [10]. We review here the definitions of those linearly ordered extensions. The intervals in a GO-space or a LOTS are written by the symbols [a,b], [a,b[,]a,b] and]a,b[. For example, $[a,b] = \{x : a \leq x \leq b\}, [a,b[= \{x : a \leq x < b\}$ and so on. For a GO-space X, we set $R = \{x \in X : [x, \rightarrow [\in \tau - \lambda\}$ and $L = \{x \in X :] \leftarrow, x] \in \tau - \lambda\}$, where λ denotes the order topology as mentioned above. Then X^* is defined as follows:

$$X^* = (X \times \{0\}) \cup \{(x,k) : x \in R, k < 0, k \in \mathbb{Z}\} \cup \{(x,k) : x \in L, k > 0, k \in \mathbb{Z}\} \subset X \times \mathbb{Z},$$

where $\mathbb{Z}$ denotes the set of integers. On the other hand, $L(X)$ is defined as follows:

$$L(X) = (X \times \{0\}) \cup \{(x,-1) : x \in R\} \cup \{(x,1) : x \in L\} \subset X \times \{-1,0,1\}.$$

X^* and $L(X)$ are linearly ordered topological spaces equipped with the lexicographic order topologies. It is esily seen that X^* contains X as a closed subspace and $L(X)$ contains X as a dense subspace. See [8], [10] for the related topics of X^* and $L(X)$. In both cases, X and $X \times \{0\}$ are identified by the correspondence of x to $(x,0)$.

See [5] and [7] for the studies on S_δ-diagonals and dense S_δ-diagonals of generalized ordered spaces and their linear extensions.

3 Separable spaces and linearly ordered extensions

Definition 3.1 Let X be a topological space. X is *separable* or a *separable space* if X has a countable dense subset.

Theorem 3.2 *If X is a separable GO-space, then we have*
(1) X is first countable.
(2) X is weakly S-normal.

Proof. (1) Let D be a countable dense subset of X, say $D = \{d_i : i \in N\}$. Let $x \in R$ be a point, where R is defined in Section 2. Take a decreasing sequence $\{d_{i_k} : k \in N\} (\subset D)$

that converges to x. Then a collection $\{[x, d_{i_k}[: k \in N\}$ of countably many open subsets builds a neighborhood base at x. For $x \in L$, we take an increasing sequence $\{d_{j_k} : k \in N\}$ that converges to x. Then a collection $\{]d_{j_k}, x] : k \in N\}$ of open subsets becomes a neighborhood base at x. Similarly, a collection $\{]d_{j_k}, d_{i_k}[: k \in N\}$ of open subsets of X is a neighborhood base at $x \in X \setminus (R \cup L)$. (2) Let C be a closed subset of X. Since C is separable by [9], C is first countable by (1) and there exists a countable dense subset D_C of C. It is easy to see that D_C is an S_δ-subset of X. Hence X is a weakly S-normal space.

Remark 3.3 The Michael line is first countable and (weakly) S-normal, but not separable. Therefore, a GO-space satisfying the conditions (1) and (2) does not imply a separable space.

Theorem 3.4 *Suppose that X is a GO-space and that $R \cup L$ is a countabe set. If X is separable, then so is X^*.*

Proof. Since X is a separable space, there exists a countable dense subset D of X. Then $D \cup R \times \{-n : n \in N\} \cup L \times \{n : n \in N\}$ is a countable dense subset of X^*.

Remark 3.5 Theorem 3.4 does not hold without the countability of $R \cup L$. To see this, consider the Sorgenfrey line S. A dense subset of S^* must contain $S^* \setminus (S \times \{0\})$, because the set $S^* \setminus (S \times \{0\})$ is open. Therefore, a dense subset is not countable.

Theorem 3.6 *Let X be a generalized ordered space. If X is separable, so is $L(X)$.*

Proof. Since X is a dense subspace of $L(X)$, it is easy to prove the theorem.

4 Weakly S-normal GO-spaces and linearly ordered extensions

Theorem 4.1 *Suppose that X is a generalized ordered space and that $R \cup L$ is a countable set. If X is weakly S-normal, then so is X^*.*

This theorem follows from the following, because a weakly S-normal GO-space is first countable by Proposition 1.6 and an S-normal space is weakly S-normal by Proposition 1.7.

Theorem 4.2 *Let X be a first countable generalized ordered space. If $R \cup L$ is a countable set, then X^* is an S-normal space.*

Proof. Set $R \cup L = \{d_i : i \in N\}$. Let $d_i \in R$. Since X is first countable, there exists a decreasing sequence $\{x_k : k \in N\}$ in X that converges to d_i. Take an interval $U(d_i, k) = [d_i, x_k[$ in X^*. Analogously, for each $d_j \in L$, take an interval $V(d_j, k) =]y_k, d_j]$ in X^*, where $\{y_k : k \in N\}$ is an increasing sequence in X that converges to d_j. If $x \in X \setminus (R \cup L)$, then there exist two such sequences $\{x_k : k \in N\}$ and $\{y_k : k \in N\}$ that converge to x. Take an interval $W(x, k) =]y_k, x_k[$ in X^*. Then we have a collection $\mathcal{C} = \{U(d_i, k) : d_i \in R, k \in N\} \cup \{V(d_j, k) : d_j \in L, k \in N\} \cup \{W(x, k) : x \in X \setminus (R \cup L), k \in N\}$ of countably many open subsets of X^*. Let C be a closed subset of X^*. Then $\mathcal{C} \cup \{C \setminus X\}$ is a required collection to assure C an S_δ-subset of X^*.

5 Examples

Example 5.1 In the case of $L(X)$, the situation is different from Theorem 4.1. To show this, let X be a GO-space obtained from $[0, \omega_1]$ by isolating ω_1. Then X is a weakly S-normal space, but $L(X)$ is not weakly S-normal although R is a one-point set. Note that $L(X)$ is homeomorphic to $[0, \omega_1] \cup \{(\omega_1, 0)\}$, where the point $(\omega, -1)$ in $L(X)$ is identified with ω_1 in $[0, \omega_1]$.

Example 5.2 Theorem 4.2 does not hold without the additional condition of countability of $R \cup L$. As is already pointed out, S^* is not S-normal, where $R = S$ (the set of all points) is not countable.

Example 5.3 Let X be a weakly S-normal GO-space. Even though $R \cup L$ is a countable set, X is not necessarily S-normal. Let I^2(lex) be a unit square (LOTS) with the usual lexicographic order. Then I^2(lex) is a weakly S-normal space and $R \cup L = \phi$, but not S-normal since a closed subset $I \times \{0, 1\}$ can not be an S_δ-subset of I^2(lex).

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